

Emission of gravitational radiation from ultra-relativistic sources

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Abstract

Recent observations suggest that blobs of matter are ejected with ultra-relativistic speeds in various astrophysical phenomena such as supernova explosions, quasars, and microquasars. In this paper we analyze the gravitational radiation emitted when such an ultra-relativistic blob is ejected from a massive object. We express the gravitational wave by the metric perturbation in the transverse-traceless gauge, and calculate its amplitude and angular dependence. We find that in the ultra-relativistic limit the gravitational wave has a wide angular distribution, like $1 + \cos \theta$. The typical burst's frequency is Doppler shifted, with the blue-shift factor being strongly beamed in the forward direction. As a consequence, the energy flux carried by the gravitational radiation is beamed. In the second part of the paper we estimate the anticipated detection rate of such bursts by a gravitational-wave detector, for blobs ejected in supernova explosions. Dar and De Rujula recently proposed that ultra-relativistic blobs ejected from the central core in supernova explosions constitute the source of Gamma-ray bursts. Substituting the most likely values of the parameters as suggested by their model, we obtain an estimated detection rate of about 1 per year by the advanced LIGO-II detector.

I. INTRODUCTION

Relativistic jets seem to be emitted by astrophysical systems wherein mass is accreted at high rate from a disk to a central compact object (for a review see [1]). Astrophysical observations suggest that blobs of plasma are ejected with ultra-relativistic velocities in supernova explosions [2], microquasars GRS1915+105 [1], [3], [4], [5] and GRO J165-40 [6], and in active galactic nuclei hosting a massive black hole.

Recently Dar and De Rujula proposed a new model for the origin of Gamma ray bursts, the *cannonball model* [7], in which the bursts are sourced by ultra-relativistic blobs of matter emitted in supernova explosions. According to this model, in a typical supernova explosion some portion of the expanding mass falls back on the central core and forms an accretion disc. Accreted matter is then ejected as blobs of plasma in the polar directions, with a large Lorentz factor γ of order 10^3 . A strongly beamed burst of Gamma ray is created when such a blob hits a shell of expanding matter ejected earlier in the supernova explosion process. According to the cannonball model, in a typical supernova explosion a few such ultra-relativistic blobs are ejected in each of the two polar directions, yielding a Gamma ray burst made of a few strong peaks.

In this paper we investigate the gravitational radiation from such ultra-relativistic blobs of matter, and evaluate the anticipated rate of detections by the gravitational-wave detector LIGO. Since the motion is relativistic the quadrupole formula cannot be used in this problem. Instead, we solve the linearized Einstein equations using the Lienard-Wiechert formula (generalized to the gravitational case).

Gravitational radiation is emitted whenever the blob changes its velocity (the gravitational field involved in a motion with constant velocity is non-radiative). In this paper we focus on the radiation emitted when the blob is ejected from the central object and is accelerated to a large Lorentz factor. A burst of gravitational radiation is also emitted when the blob hits the ejecta, but this burst appears to be weaker by several orders of magnitude and we shall not consider it here.

The analysis throughout this paper is essentially free of assumptions about the values of the astrophysical parameters involved. We merely assume that (i) a blob of matter is ejected from a massive object (a "star") and is accelerated to a Lorenz factor $\gamma \gg 1$, and (ii) the blob's energy γm is small compared to the star's mass M . (Both assumptions $\gamma \gg 1$ and $\gamma m \ll M$ are not necessary for the analysis, but they significantly simplify it.) We obtain a general expression for the amplitude of the gravitational wave as a function of direction. We also calculate the directional dependence of the gravitational wave's observed frequency. These expressions involve two astrophysical parameters: the blob's energy γm (for the amplitude), and the typical time scale Δt of acceleration (for the frequency). We then derive a general expression for the anticipated detection rate. This expression depends on three more parameters: the event rate per unit volume (e.g. the event of supernova explosions), the detector's sensitivity, and the detector's optimal frequency. Substituting the most likely values of the astrophysical parameters, as suggested by the cannonball model, we obtain an anticipated detection rate of about 1 per year by the advanced LIGO-II detector. This detection rate is not certain, however, because of an uncertainty in the astrophysical parameters involved. In particular, the detection rate is proportional to a third power of the blob's energy, and the uncertainty in the latter may change the detection rate by one or two orders of magnitude.

Our analysis shows that despite the large Lorenz factor, the gravitational field, expressed in terms of the metric perturbation, is not strongly beamed in the forward direction. Rather, at the ultra-relativistic limit the directional dependence of the transverse-traceless (TT) metric perturbation is like $1 + \cos \theta$, where θ is the angle between the particle's velocity and the spatial direction vector from the source point to the observer. Thus, whereas the Gamma-ray burst can only be observed in a very small solid angle comparable to γ^{-2} (due to the strong beaming of the electromagnetic radiation), the gravitational signal may be observed in a wide solid angle, effectively $\sim 2\pi$ steradians (unless Δt is too large – see section VI). On the other hand, the observed frequency is strongly Doppler blue-shifted in the forward direction. As a consequence, the energy flux carried by the gravitational waves

is beamed in the forward direction.

The gravitational radiation emitted in this process has two special features which distinguish it from most other sources. First, we are dealing here with a "burst with memory" [8]. That is, at the end of the process (e.g. ejection of a blob) the metric perturbation amplitude does not return to its original value (see section IV). Secondly, according to the cannonball model, in a typical supernova event several blobs are emitted (in each of the two polar directions). Consequently, the gravitational signal will be composed of a few separate bursts. Therefore, once such an event is detected, it may be easy to distinguish it from other sources of gravitational radiation.

In section II we calculate the metric perturbation produced by the moving blob. We first calculate it in the Lorenz gauge, using the gravitational analog of the Lienard-Wiechert formula. Then we transform the metric perturbation to the TT gauge. In section III we obtain the angular dependence of the wave's amplitude in the ultra-relativistic limit. We find this amplitude to be proportional to $1 + \cos \theta$ (with a narrow "hole" at the center, whose angular width is $\sim 1/\gamma$). We also discuss the relation between our results and a previous work by Dray and t'Hooft (DtH) [9], who analyzed the gravitational field of a particle moving at a (fixed) ultra-relativistic speed. Then in section IV we calculate the total change in the metric perturbation that occurs when a massive star emits an ultra-relativistic blob. We show that this change is non-vanishing (namely, this is a "burst with memory"). Furthermore, this change is (at the leading order) equal to the contribution of the blob itself to the metric perturbation. In section V we obtain the angular dependence of the observed burst's frequency, which is strongly Doppler blue-shifted in the forward direction. We show that unlike the metric perturbation, the energy flux is indeed beamed in the forward direction, $\theta \sim 1/\gamma$.

In section VI we derive the expression for the anticipated detection rate, as a function of the various parameters involved. Finally, in section VII we substitute in this general expression the astrophysical parameters emerging from the cannonball model, as well as the LIGO-II detector's parameters. Of these parameters, two have the largest uncertainty: the

blob's energy $E = \gamma mc^2$ and the acceleration time Δt . The cannonball model yields an order-of-magnitude estimate for the blob's energy: $E \sim 10^{52} \text{erg}$. The parameter Δt has a larger uncertainty; however, this parameter does not affect the detection rate as long as it is smaller than the detector's typical time scale. And, even if it is larger, it only affects the detection rate through its first (inverse) power. Therefore, the main uncertainty seems to come from the energy parameter, which enters the detection rate as E^3 . With the above value for E , we obtain a detection rate of about 1 event per year in the advanced LIGO-II detector (provided that Δt is not too large).

For the above value of E , the maximal distance for observation by LIGO-II is found to be $R_{\text{max}} \sim 15 \text{Mpc}$. Since this corresponds to $z \ll 1$, we ignore the cosmological redshift effects throughout the paper.

Several authors, mostly during the 1970's, investigated the gravitational radiation emitted from ultra-relativistic sources ([10], [11], [12], [13], [14], [15], [16] and references therein). These authors considered a variety of model problems, used several methods of calculation, and studied various aspects of the gravitational field emitted. We haven't found any previous work which covers the problem that concerns us here. The closest we have found is the analysis by Adler and Zeks [12], who considered a similar problem of a supernova explosion. However, they calculated the TT wave-form only in the special case of two equal masses, whereas the astrophysical situation that concerns us here is $m \ll M$. (Indeed, in the special case of two equal-mass blobs ejected simultaneously in the two polar directions, our result agrees with Ref. [12] - see section IV.) Peters [10], and later Kovacs and Thorne [13], [15], considered ultra-relativistic encounters, but their analysis is restricted to the case of a large impact parameter, whereas in our problem the impact parameter vanishes. Ruffini [11] Smarr [14] and D'earth [16] studied the ultra-relativistic head-on collision of two black holes (or of a small object with a massive black hole [11]). This situation differs from our case, in which the star and the blob are both weak-field objects. (There also is a difference in the time direction, i.e. an explosion instead of a collision, but at least in our star-blob system the gravitational radiation does not care about this change in the direction of time - see

section V). Naively one might expect that the gravitational waves produced in the collision will not be sensitive to the nature of the objects involved, as long as the latters are small. Our analysis, however, suggest the contrary for the head-on case. In our star-blob problem we find the energy flux of gravitational radiation to be beamed in the forward direction (see section V). No such beaming occurs in the head-on collision of two black holes [14], [16]. This difference between the two problems has a simple intuitive reason, which we discuss in section V.

We use the signature $(-+++)$. Since most of the paper deals with basic general-relativistic analysis, we use general-relativistic units $G = C = 1$. Only in section VII, in which we put astrophysical numbers, we retain the values of G and C in standard physical units.

II. GRAVITATIONAL FIELD OF A RELATIVISTIC PARTICLE

For a weak gravitational field, the linearized Einstein equations (expressed in Cartesian coordinates x^α) read [17]

$$16\pi T_{\mu\nu} = -\bar{h}_{\mu\nu,\alpha}{}^\alpha - \eta_{\mu\nu} \bar{h}_{\alpha\beta}{}^{\alpha\beta} + \bar{h}_{\mu\alpha,\nu}{}^\alpha + \bar{h}_{\nu\alpha,\mu}{}^\alpha, \quad (1)$$

where $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\lambda{}_\lambda$ and $h_{\mu\nu}$ is the metric perturbation. Under the Lorenz gauge conditions $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$, equation (1) reduces to

$$\bar{h}_{\mu\nu,\alpha}{}^\alpha = -16\pi T_{\mu\nu}. \quad (2)$$

Consider a point mass m (a "particle") moving along a world line $r^\alpha(\tau)$, where τ is the proper time and r^α denotes the particle's location in Cartesian coordinates. The energy momentum tensor of such a point mass is given by

$$T^{\alpha\beta}(x) = m \int u^\alpha(\tau) u^\beta(\tau) \delta^{(4)}[x - r(\tau)] d\tau, \quad (3)$$

where $u^\alpha = dr^\alpha/d\tau$ is the particle's 4-velocity. The retarded solution of Equation (2) for such a source term is obtained by a straightforward generalization of the Lienard-Wiechert potentials:

$$\bar{h}^{\alpha\beta}(x) = 4m \frac{u^\alpha(\tau) u^\beta(\tau)}{-u_\gamma \cdot [x - r(\tau)]^\gamma} \Big|_{\tau=\tau_0} . \quad (4)$$

This expression is to be evaluated at the retarded time τ_0 , which is the intersection time of $r^\alpha(\tau)$ and the observer's past light-cone. The metric perturbation in the Lorenz gauge is then given by

$$h^{\alpha\beta} = \bar{h}^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} \bar{h}_\gamma^\gamma = \frac{4m}{-u_\gamma \cdot [x - r(\tau)]^\gamma} \left[u^\alpha(\tau) u^\beta(\tau) + \frac{1}{2} \eta^{\alpha\beta} \right] . \quad (5)$$

In the next step we transform $h_{\mu\nu}$ from the Lorenz gauge to the TT gauge, which is best suited for calculating the response of a gravitational-wave detector. The metric perturbation in the TT gauge, which we denote $h_{\mu\nu}^{TT}$, includes only space-space components, namely $h_{t\mu}^{TT} = 0$. This spatial part is obtained from $h_{\mu\nu}$ by [17]

$$h^{TT} = PhP - \frac{1}{2} P \cdot \text{Tr}(hP) . \quad (6)$$

Here h^{TT} , h , and P are 3×3 spatial matrices, where h represents the spatial part of $h_{\mu\nu}$, and P is a projection operator defined by $P_{jk} = \delta_{jk} - \hat{n}_j \hat{n}_k$, where \hat{n} is the unit spatial direction vector from the (retarded) source point to the observer. Hereafter Latin indices run over the three spatial components. [In Eq. (6) we have omitted the indices for brevity, and we use the standard matrix product notation].

Before proceeding with the calculation, there is a subtlety that must be addressed. The projection operation (6) applied to a metric perturbation h constitutes a gauge transformation only if h is a pure gravitational-radiation field. In our case, for a particle moving at a fixed speed, h is non-radiative. Despite of this, the application of Eq. (6) to our problem is justified, because of the following reason: The physical quantity that will concern us in this paper is not the value of h , but rather the *change* in h that occurs during an astrophysical process. This change, which we denote Δh , occurs when the particle changes its velocity due to interaction with another object. The quantity Δh represents a pure radiation field, and hence applying the projection (6) to it yields a valid gauge transformation. It is this radiative piece Δh (in fact, its TT part) that is relevant for detection over astrophysical

distances. Obviously, in order to have a nonvanishing Δh we must consider a system of two particles (or more), interacting with each other. Then we have to sum over the contributions to h from all components of the system. Correspondingly, the quantity relevant for gravitational wave detection is the change in this sum, namely $\Delta h = \Delta(\Sigma h)$, where Σ denotes a summation over the components of the system. This quantity is a pure radiation field, and from the linearity of the projection (6), the TT-part of Δh is given by

$$\Delta h^{TT} \equiv (\Delta h)^{TT} = \Delta(\Sigma(h^{TT})) . \quad (7)$$

In what follows we shall calculate h^{TT} (for a single object), and in section IV we shall construct from it the quantity Δh^{TT} for the situation of a blob ejected from a star. (In fact, we shall show that at the leading order Δh^{TT} is nothing but h^{TT} of the ejected blob.)

Proceeding with the calculation of h^{TT} , one finds that the term proportional to $\eta^{\alpha\beta}$ in Eq. (5), which represents a pure trace term, vanishes upon the projection (6), therefore

$$h^{TT} = P\bar{h}P - \frac{1}{2}P \cdot \text{Tr}(\bar{h}P) , \quad (8)$$

where \bar{h} denotes the 3×3 spatial part of $\bar{h}_{\mu\nu}$. It is useful to decompose this expression into an amplitude factor and a directional factor. We define the particle's 3-velocity, $\bar{v} = d\bar{r}/dt$, the particle's speed $v = |\bar{v}|$, and the unit 3-vector in the velocity direction, $\hat{v} = \bar{v}/v$ (hereafter a bar denotes a spatial 3-vector, and a unit 3-vector is denoted by a hat). Defining $w_{ij} \equiv \hat{v}_i \hat{v}_j$, we find $\bar{h}_{ij} = h_0 w_{ij}$ where

$$h_0 = \frac{4m(\beta\gamma)^2}{-u_\gamma \cdot [x - r(\tau)]^\gamma} = \frac{4\gamma m\beta^2}{R(1 - \beta \cos \theta)} . \quad (9)$$

Here β and γ denote the standard special-relativistic quantities $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$, R is the spatial distance between the evaluation point x^α and the source point r^α , and θ is the angle between \hat{n} and \hat{v} , i.e. $\hat{n} \cdot \hat{v} = \cos \theta$. Equation (8) now reads

$$h^{TT} = h_0 \left[PwP - \frac{1}{2}P \cdot \text{Tr}(wP) \right] = h_0 \left[PwP - \frac{1}{2}P \sin^2 \theta \right] . \quad (10)$$

This expression enfolds the information on both the amplitude and the polarization of the emitted gravitational wave. The direction of polarization is dictated by the transversality

condition and the direction of motion. Thus, at a given space-time point, the maximal detection amplitude, which we denote h_+ , is achieved for a detector's arm directed perpendicular to \hat{n} in the \hat{n} - \hat{v} plane. The same amplitude (but with opposite sign) is obtained in the perpendicular transverse direction, i.e. in the direction perpendicular to both \hat{n} and \hat{v} , which we denote \hat{n}^+ . The gravitational wave's amplitude h_+ can thus be obtained by projecting h^{TT} on the direction \hat{n}^+ :

$$h_+ = -\hat{n}_i^+ h_{ij}^{TT} \hat{n}_j^+ . \quad (11)$$

(Hereafter, a repeated spatial sub-index denotes a summation.) Substituting Eq. (10) in (11), we encounter two types of directional terms: $\hat{n}_i^+ (PwP)_{ij} \hat{n}_j^+$ and $\hat{n}_i^+ P_{ij} \hat{n}_j^+$. The former is nothing but the square of $\hat{n}_i^+ P_{ij} \hat{v}_j$. One immediately verifies that $\hat{n}_i^+ P_{ij} \hat{n}_j^+ = 1$ and $\hat{n}_i^+ P_{ij} \hat{v}_j = 0$, and therefore

$$h_+ = (1/2)h_0 \sin^2 \theta = \frac{2\gamma m \beta^2}{R} \frac{\sin^2 \theta}{1 - \beta \cos \theta} . \quad (12)$$

It is illuminating to compare this expression for gravitational perturbations to that of the electromagnetic or scalar field of a point source in motion. For a scalar field ϕ and an electromagnetic four-potential A^α in the Lorenz gauge, the standard Lienard-Wiechert solution yields (e.g. [18], [19]):

$$\phi = q \frac{1}{-u_\gamma \cdot [x - r(\tau)]^\gamma} \Big|_{\tau=\tau_0} , \quad A^\alpha = q \frac{u^\alpha(\tau)}{-u_\gamma \cdot [x - r(\tau)]^\gamma} \Big|_{\tau=\tau_0} , \quad (13)$$

where q denotes the scalar or electric charge, respectively. Let us denote by A_α^T the four-potential in the transverse gauge. The temporal component A_0^T vanishes, and the spatial part is given by $A_i^T = P_{ij} A_j$. Let A^T denote the magnitude of A_i^T , i.e. $A^T = (A_i^T A_i^T)^{1/2}$. Then a straightforward calculation yields

$$A^T = \frac{q}{R} \frac{\beta \sin \theta}{1 - \beta \cos \theta} . \quad (14)$$

To represent all three cases in a single equation, let ψ_s denote ϕ , A^T , or h_+ , for $s = 0, 1, 2$, respectively. Then

$$\psi_s = s! \frac{q}{R} \frac{(\gamma\beta \sin \theta)^s}{\gamma(1 - \beta \cos \theta)} , \quad (15)$$

where in the gravitational case ($s = 2$) q denotes the particle's mass m .

III. ANGULAR DISTRIBUTION IN THE ULTRA-RELATIVISTIC CASE

We shall now consider the angular distribution of the gravitational field in the limit $\gamma \gg 1$, i.e. $\beta \cong 1$. We can then omit the factor β^2 in Eq. (12):

$$h_+ \cong \frac{2\gamma m}{R} \frac{\sin^2 \theta}{1 - \beta \cos \theta} \quad (\gamma \gg 1) . \quad (16)$$

However, the factor β in the denominator must be treated more carefully. For $\gamma \gg 1$ we can always approximate

$$1 - \beta \cos \theta = (1 - \beta) + \beta(1 - \cos \theta) \cong (\gamma^{-2}/2) + (1 - \cos \theta) . \quad (17)$$

Now, for $\theta \ll 1$ we can approximate $\cos \theta \cong 1 - \theta^2/2$, hence

$$1 - \beta \cos \theta \cong (\gamma^{-2} + \theta^2)/2 \quad (\theta \ll 1) . \quad (18)$$

On the other hand, for θ large compared to $1/\gamma$ we can ignore the first term in the right-hand side of Eq. (17):

$$1 - \beta \cos \theta \cong 1 - \cos \theta \quad (\theta \gg \gamma^{-1}) . \quad (19)$$

Thus, in the ultra-relativistic limit h_+ takes two qualitatively different asymptotic forms, depending on the value of θ . For small θ we have

$$h_+ \cong \frac{4\gamma m}{R} \frac{\theta^2}{\gamma^{-2} + \theta^2} \quad (\theta \ll 1) , \quad (20)$$

and for θ large compared to $1/\gamma$,

$$h_+ \cong \frac{2\gamma m}{R} (1 + \cos \theta) \quad (\theta \gg \gamma^{-1}) . \quad (21)$$

The two asymptotic regions overlap at $\gamma^{-1} \ll \theta \ll 1$, where we have $h_+ \cong 4\gamma m/R$. In the range of very small angles, $\theta \lesssim \gamma^{-1}$, there is a "hole" in the angular distribution, wherein h_+ sharply decreases and vanishes at $\theta = 0$.

Motivated by the beaming phenomenon in the analogous electromagnetic problem, we shall refer to the regions $\theta \lesssim \gamma^{-1}$ and $\theta \gg \gamma^{-1}$ as the beaming and off-beaming zones, respectively. Note, however, that the metric perturbation h_+ does not exhibit a true beaming phenomenon. There is no sharp enhancement of h_+ in the narrow forward direction where $\theta \sim \gamma^{-1}$; rather, there is a "hole" at $\theta < \gamma^{-1}$. (In the off-beaming zone there is a slow increase in h_+ when θ decreases, like $1 + \cos \theta$, but this is a moderate, γ -independent, increase.)¹

In the ultra-relativistic case, which concerns us here, the beaming zone covers an extremely small solid angle, $\Omega \propto \gamma^{-2}$. Since there is no enhancement of h_+ in this range, this zone has a negligible contribution to the anticipated rate of detection. Therefore, in what follows we shall ignore the "hole" in the beaming zone, and always use the off-beaming expression (21) for h_+ . It is remarkable that the perturbation amplitude (21) does not depend on m or on the particle's speed separately. It only depends on the product γm , i.e. on the particle's energy.

DtH [9] analyzed the gravitational field sourced by an ultra-relativistic particle moving along a geodesic in flat space. They found that in the limit $\beta \rightarrow 1$ the gravitational field forms a planar shock wave which propagates along with the particle, at the speed of light. This behavior is very different from what we have found here. This difference cannot be explained by the choice of different gauges; For example, in the DtH analysis the geometry before *and behind* the shock wave is strictly flat. The reason for this difference is that, the two analyses consider different limiting procedures. DtH took the (constant-speed) boosted Schwarzschild

¹ This qualitative picture changes when one describes the gravitational radiation in terms of the Riemann tensor or the effective energy-momentum outflux. In both descriptions, there is a beaming effect at angles $\theta \sim \gamma^{-1}$; See section V for the energy flux.

solution, evaluated the gravitational field at a fixed, finite R , and then took the limit $\beta \rightarrow 1$. We are applying here a different limit, which is better adopted to the astrophysical situation that motivates the present work: We assume that the particle's speed changes with time, then pick the radiative piece of the gravitational field associated with this change of velocity, and only then we take the limit $\beta \rightarrow 1$. In this procedure, the non-radiative piece of the gravitational field is dropped. (Recall that the TT-projection only respects the radiative part of the gravitational field.) This non-radiative piece is unimportant for detection over astrophysical distances; yet it is the only piece which exists in the DtH problem, in which the gravitational field is non-radiative.

Let us consider a situation which is perhaps not too realistic from the astrophysical point of view, but it may clarify the relation between the two analyses. Assume that a blob is ejected from the star at $t = 0$, with a finite Lorenz factor $\gamma \gg 1$, and the observer is located at a large R and a very small θ . The blob then moves undisturbed along a geodesic all the way from the star to the observer's neighborhood, and passes near the observer. Then we expect that, provided that the blob's minimal distance to the observer is small enough (i.e. θ is sufficiently small), as the blob passes by, the observer will watch a wave phenomenon similar to that described by DtH (though the shock will be somewhat smoothed, due to the finite γ). In addition, the observer will also watch the radiative phenomenon associated with the ejection, described by the gravitational field (21). There will be a time lag between the two phenomena: The ejection-induced radiation pulse will arrive at the moment $t = R$, corresponding to zero retarded time (we neglect here the small quantity Δt). On the other hand, since the shock-like wave must move along with the particle, it will reach the observer at the moment $t \cong R + (R/2)(\gamma^{-2} - \theta^2)$. The two waves will also have different frequencies: The typical frequency of the DtH wave will depend on θ and γ , but will also decrease like $1/R$. The radiative field (21) will have an R -independent typical frequency (which for small θ is $\sim \gamma^2/\Delta t$; see section V). Therefore, for a detection over astrophysical distances the ejection-induced wave seems to be the more important phenomenon.

The DtH phenomenon will only take place in the range $\theta < 1/\gamma$, because for $\theta > 1/\gamma$ the

time lag becomes negative, which would contradict causality. As θ increases in the range $0 < \theta < 1/\gamma$, the DtH phenomenon will become weaker (because the minimal blob-observer distance increases with θ) and the radiative field will become stronger [cf. Eq. (20)]. The relation between these two gravitational-wave phenomena and their possible coexistence deserve further investigation.

IV. SHOOTING AN ULTRA-RELATIVISTIC BLOB

We shall now calculate Δh^{TT} , i.e. the change in h (expressed in the TT gauge) that occurs when a star ejects an ultra-relativistic blob of matter. As was discussed in section II, it is this quantity which is relevant for analyzing the detector's response. We shall show that for a sufficiently small blob's energy, this change in the overall metric perturbation coincides with the contribution h^{TT} of the blob.

Consider a star with mass M initially at rest. At a given moment $t = t_0$ it emits a blob of mass $m \ll M$ with an ultra-relativistic speed $\gamma \gg 1$. Let us assume that the whole acceleration process ends at $t = t_1 \equiv t_0 + \Delta t$ (these times all refer to the star's rest frame). The mechanism responsible for this process is unimportant for this discussion (it could be, for example, an electromagnetic acceleration due to a dynamo effect or some MHD instability accelerating a blob of plasma, or radiation pressure). The important point is that at $t < t_0$ the entire system can be modeled as being at rest, then the blob accelerates between $t = t_0$ and $t = t_1$, and at $t > t_1$ the blob moves with a constant speed, $\gamma \gg 1$. (One can think of this process as the time-reversal of a fully-inelastic collision.) Let us denote the remaining mass of the star by M_1 . We assume that the blob's energy γm is $\ll M$. Then from energy-momentum conservation one finds that, at the leading order in the small parameter $\gamma m/M$, $M_1 \cong M - \gamma m$ and the star's speed is non-relativistic, $\beta_s \cong -\gamma m/M$.²

² In principle one has to include in this energy-momentum balance the amount of energy E_g (and also the momentum) carried by the gravitational radiation. However, E_g is quadratic in γm ; One

Since all motions in this process are along the same axis, the polarization of both the blob's and the star's fields will have the same direction. The TT metric perturbation at the end of the process can therefore be described in terms of the overall amplitude parameter h_+ (obtained by summing the amplitudes h_+ of the star and the blob). Correspondingly, the change in h^{TT} is given by the quantity $\Delta h_+ = h_+^1 - h_+^0$, where h_+^0 and h_+^1 denote the overall field amplitude before and after the process, respectively. Since initially $\beta = 0$, from Eq. (12) we have $h_+^0 = 0$. At $t > t_1$ we have $h_+^1 = h_+^{star} + h_+^{blob}$, where

$$h_+^{star} \cong \frac{2M\beta_s^2}{R} \sin^2 \theta \quad , \quad h_+^{blob} \cong \frac{2\gamma m}{R} (1 + \cos \theta) \quad . \quad (22)$$

The ratio of these two contributions is

$$h_+^{star}/h_+^{blob} \cong \frac{M\beta_s^2}{\gamma m} \frac{\sin^2 \theta}{1 + \cos \theta} \cong 2 \sin^2(\theta/2) \frac{\gamma m}{M} << 1 \quad , \quad (23)$$

hence the star's contribution to h_+^1 is negligible. We conclude that the change in h_+ is just the contribution of the ultra-relativistic blob:

$$\Delta h_+ = h_+^1 \cong h_+^{blob} \cong \frac{2\gamma m}{R} (1 + \cos \theta) \quad . \quad (24)$$

The fact that Δh_+ does not vanish means that we are dealing here with a "burst with memory" [8]. One may be puzzled by this lack of conservation in the overall perturbation field, because the source term, the particles' energy-momentum tensor, is conserved. The resolution of this puzzle is simple: The Lienard-Wiechert solution (4) has a dependence on the velocity also through the denominator, $u_\gamma [x - r(\tau)]^\gamma$. It is this dependence which leads to the non-vanishing of Δh_+ .

In several astrophysical systems with an accretion disc, the blobs appear to be emitted in pairs, along the two polar directions. In such a case, h_+ will simply be the sum of the contributions from the two blobs. Note that the contributions from the two blobs do not

can show that $E_g/\gamma m < \gamma m/M << 1$, so E_g may be ignored. The momentum carried by the gravitational radiation is also bounded by E_g .

cancel each other. For example, for a symmetric pair of ultra-relativistic blobs, each carrying energy γm , the sum of the two contributions will be direction-independent:

$$\Delta h_+ = h_+^1 \cong \frac{4\gamma m}{R} . \quad (25)$$

This result agrees with the analysis by Adler and Zeks [12], who considered the case of two equal masses [20]. (From the observational point of view, however, recall that the two components may arrive the detector at different times, and will also have different Doppler factors.)

V. ANGULAR DEPENDENCE OF OBSERVED BURST FREQUENCY

Let f_c denote the characteristic frequency of the observed burst. It is given by $f_c \sim \delta t^{-1}$, where δt denotes the burst duration (i.e. the rising time of h_+ from its initial to final value) as measured by the detector. We need to relate δt to the pulse duration in the star's Lorenz frame, $\Delta t = t_1 - t_0$. Let $t'(t)$ denote the arrival time (at the detector) of a null geodesic which emerges from the blob at a moment t . These two times are related by $t' = t + R$. A straightforward calculation then yields

$$\frac{dt'}{dt} = 1 - \beta \cos \theta . \quad (26)$$

As was discussed in section III, the beaming zone is extremely narrow and does not significantly contribute to the detection rate. We shall therefore ignore it and use the off-beaming approximation (19):

$$\frac{dt'}{dt} \cong 1 - \cos \theta , \quad (27)$$

and hence $\delta t \cong (1 - \cos \theta) \Delta t$. Therefore the observed frequency is

$$f_c \sim [(1 - \cos \theta) \Delta t]^{-1} . \quad (28)$$

This relation is independent of γ . Recall, however, that this approximation breaks at $\theta \lesssim \gamma^{-1}$: At the beaming zone f_c saturates at a maximal value $f_c^{\max} \sim \gamma^2 / \Delta t$.

Although there is no enhancement of h in the beaming zone, the energy flux is in fact beamed in the forward direction, as we now show. Let F denote the time-integrated energy flux per unit solid angle. Since the energy density is proportional to $h^2\omega^2 \sim h^2/\delta t^2$, we have $F \sim R^2 h^2/\delta t \sim (\gamma m)^2/\delta t$. Thus, outside the beaming zone (but at $\theta \ll 1$) F behaves as $[(\gamma m)^2 \Delta t^{-1}] \theta^{-2}$, and it gets a maximal value of order $\gamma^2(\gamma m)^2 \Delta t^{-1}$ at $\theta \sim 1/\gamma$.

This result is remarkable, because no such beaming occurs in the analogous situation of an ultra-relativistic head-on collision of two black holes [14], [16]. To sharpen the contrast between the two cases, consider the time-reversal variant of our problem, i.e. a blob colliding fully inelastically with a star. As one can easily verify, the pattern of Δh in this collision problem will be exactly the same as in the original ejection problem – and, in particular, F will be beamed in the forward direction. The intuitive reason for the difference between the two collision problems is simple: In our star-blob system, the gravitational field is everywhere weak; the interaction between the star and the blob is non-gravitational. In particular, the (de-)acceleration occurs on a distance scale Δt which is $\gg M$. Hence the beamed gravitational energy flux propagates to null infinity without any obstacle. On the other hand, in the analogous black-holes head-on collision problem the interaction is solely gravitational, and the (de-)acceleration occurs on a distance scale which (to the extent it is defined) is of order M . Hence, should any beamed radiation form, it would immediately be swallowed by the large black hole.³ It is therefore not surprising that no beamed radiation is observed at null infinity. [21] Indeed, in the case of an ultra-relativistic black-holes encounter with a large impact parameter, the gravitational radiation does exhibit a beaming [10], [14],

³ This argument does *not* imply that a beamed radiation will actually hit the large black hole: Since the deacceleration distance in this case is of the same order of magnitude as the radius of curvature of the large black hole, M , the qualitative flat-space arguments are not applicable for the analysis of the behavior of radiation near the black hole. In fact, the equivalence principle suggests that no beamed radiation will hit the black hole.

[15], [16].

VI. CALCULATION OF DETECTION RATE

In this section we shall evaluate the burst's detection rate, as a function of the various parameters involved. To this end, we shall first consider the detector's sensitivity. Then we calculate, for a given burst, the detection distance, the detection angular range, and the detection volume. Given the rate of such events of blob ejection, we shall derive the general expression for the detection rate.

Detector sensitivity

Let us denote the detector's peak sensitivity by h_d , and the frequency at which this maximal sensitivity is achieved by f_d . For a burst with memory, the detector will have the maximal sensitivity h_d as long as the burst's observed frequency f_c is larger than f_d [8]. The detector's sensitivity is quickly degraded at $f_c < f_d$, and in the calculation below we shall neglect this range for simplicity (this may result in a small decrease in the calculated detection rate). Thus, we shall presume that a detection occurs if the following two conditions are satisfied:

$$h_+ > h_d \tag{29}$$

and

$$f_c > f_d . \tag{30}$$

Detection distance

From condition (29) and Eq. (21), at a given direction the observation distance is

$$R_o = \frac{2\gamma m}{h_d}(1 + \cos \theta) . \tag{31}$$

The maximal observation distance is obtained at the forward direction,

$$R_{\max} = 4\gamma m/h_d . \quad (32)$$

Detection angle

From condition (30) and Eq. (28), the angular detection range is bounded by

$$(1 - \cos \theta) < (f_d \Delta t)^{-1} . \quad (33)$$

We shall distinguish between two cases:

Case A- $\Delta t < 1/(2f_d)$: In this case, the burst may be observed in all directions.

Case B- $\Delta t > 1/(2f_d)$: In this case, the burst is observed in the range $\theta < \theta_{\max}$, where θ_{\max} is given by

$$1 - \cos \theta_{\max} = (f_d \Delta t)^{-1} . \quad (34)$$

There is a third case, in which Δt is of order γ^2/f_d or larger. In this range, the observed frequency will be too small even at the forward direction. Hereafter we shall assume $\Delta t \ll \gamma^2/f_d$. (This assumption seems very reasonable if we take e.g. $\gamma \sim 10^3$; See the discussion of the values of the various parameters in the next section.) Note also that for such a large Δt , θ_{\max} of case B is so small that the resulting detection rate is negligible anyway.

Detection volume

Consider an event of a blob ejection. We shall now calculate the volume of the region of space in which the emitted burst will be detected. This volume is given by

$$V = \frac{1}{3} \int_{\theta < \theta_{\max}} R_o^3(\theta) d\Omega = \frac{2\pi}{3} \int_{\cos \theta_{\max}}^1 R_o^3(\theta) d \cos \theta . \quad (35)$$

(In case A, we take $\theta_{\max} = \pi$.) Substitution of Eq. (31) in the last expression yields

$$V = \frac{64\pi}{3} (\gamma m/h_d)^3 w , \quad (36)$$

where

$$w = 1 - (1 + \cos \theta_{\max})^4/16 . \quad (37)$$

Evaluating w in the two cases A,B, we find

$$w = \begin{cases} 1 & (\Delta t < \Delta t_d) , \\ 1 - (1 - \Delta t_d/\Delta t)^4 & (\Delta t > \Delta t_d) , \end{cases} \quad (38)$$

where $\Delta t_d \equiv 1/(2f_d)$. In the case of large Δt , we get

$$w \cong 4\Delta t_d/\Delta t \quad (\Delta t \gg \Delta t_d) . \quad (39)$$

Detection rate

Let us denote by n the rate of supernovae explosions per unit volume. We assume here that each supernova ejects blobs to the two polar directions, which adds another factor of two.⁴ The detection rate is therefore

$$N = 2nV = \frac{128\pi}{3} n (\gamma m/h_d)^3 w . \quad (40)$$

According to the cannonball model, in each of the two polar directions several blobs are emitted, typically of the order 3-5. We do not multiply $2nV$ by this number, because these are not independent detections. Rather, each observed event will be a composition of a few bursts.⁵

⁴ This factor 2 is not mathematically precise, and it may depend on $\Delta t/\Delta t_d$, but in the worst case - case A - it is 15/8, so we can well approximate it by 2.

⁵ It is assumed here that the time separation between two successive bursts in a given event will be

VII. INSERTING ASTROPHYSICAL PARAMETERS

We shall now evaluate the detection rate by substituting astrophysical numbers in the general expression, Eq. (40). The astrophysical situation concerned us here is that of a supernova explosion resulting in the ejection of ultra-relativistic blobs, as proposed by the cannonball model. First we re-write the above general expression, retaining the constants C and G :

$$N = \frac{128\pi}{3} n \left(\frac{\gamma m G}{c^2 h_d} \right)^3 w(\Delta t) , \quad (41)$$

where w is given in Eq. (38). We need to evaluate the supernova rate n , the blob's energy γm , and the characteristic acceleration time Δt .

Consider first the supernova rate n . The Shapley-Ames 'fiducial' sample of 342 galaxies within the Virgo circle [22], [23] has a mean B-band luminosity of $6.7h^{-2}10^9 L_\odot(B)$ and a supernova rate of $3.09h^2 [100yr10^{10} L_\odot(B)]^{-1}$. The luminosity density of the local universe is [24] $\rho_L = (2.0 \pm 0.4) \cdot 10^8 h L_\odot Mpc^{-3}$. Therefore we estimate the supernova density in the local universe (taking $h = 0.65$) as

$$n \simeq 1.7 \cdot 10^{-4} Yr^{-1} \cdot Mpc^{-3} . \quad (42)$$

For the blob's energy, the cannonball model suggests the typical value $E = \gamma m c^2 \approx 10^{52} erg$ [7], corresponding to γm of order $10^{31} gram$. Using this and Eq. (42) one gets

$$N \cong \left[\frac{7.5 \cdot 10^{-23}}{h_d} \right]^3 E_{52}^3 w(\Delta t) [Yr^{-1}] , \quad (43)$$

where E_{52} is the blob's energy in units of $10^{52} erg$.

large compared to Δt_d , so they will not sum up coherently (a coherent sum-up would significantly increase the detection distance and hence the detection rate). Note, however, that the presence of several bursts in a single event eases its detection: It makes it easier to distinguish the event from the noise, so it may decrease the signal-to-noise ratio required for detection.

The sensitivity curve for LIGO-II (advanced detector) may be found in e.g. Ref. [25]. The optimal value of the noise level is about $6 \cdot 10^{-24}$, at a frequency $f_d \approx 50 \text{ sec}^{-1}$. A minimal signal-to-noise ratio of about 5.5 [26] is required for a detection. [27] An averaging over all possible source polarizations yields an extra factor ~ 0.5 in the effective wave's amplitude. Combining these two factors, one obtains an effective threshold for detection which is about 11 times larger than the noise level [26], i.e. $h_d \approx 7 \cdot 10^{-23}$. This leads to

$$N \simeq 1.3 E_{52}^3 w(\Delta t) [Yr^{-1}] . \quad (44)$$

The characteristic time Δt is hard to estimate, because the mechanism responsible for the acceleration is unknown (presumably it is the same yet-unclear mechanism which forms the relativistic jets in various astrophysical systems including accretion discs around a compact object). The parameter w depends on the dimensionless parameter $\Delta t/\Delta t_d$, where for LIGO-II $\Delta t_d \simeq 10^{-2} \text{ sec}$. As long as Δt is smaller than Δt_d we have $w \cong 1$ and hence

$$N \simeq 1.3 E_{52}^3 [Yr^{-1}] \quad (\Delta t < \Delta t_d) . \quad (45)$$

For $\Delta t > \Delta t_d$, w decreases. If $\Delta t \gg \Delta t_d$ we may use the approximation (39) and the detection rate is inversely proportional to Δt :

$$N \simeq 5(\Delta t_d/\Delta t) E_{52}^3 [Yr^{-1}] \quad (\Delta t \gg \Delta t_d) . \quad (46)$$

A reasonable lower bound on Δt may be obtained by assuming that the acceleration to an ultra-relativistic speed occurs within a distance scale comparable to the neutron star's radius, $R_{ns} \approx 10 \text{ km}$, namely $\Delta t_{\min} \approx R_{ns}/c \approx 3 \cdot 10^{-5} \text{ sec}$.

Dar [28] suggested that the typical time scale for the whole blob ejection process should be comparable to the dynamical gravitational time-scale, e.g. the free-fall time at the neutron star's radius. This time scale is of order $\Delta t_{\text{free-fall}} \sim 10^{-4} \text{ sec}$. According to this suggestion we can safely use the detection rate (45). Since $\Delta t_{\text{free-fall}} \ll \Delta t_d$, this conclusion will hold even if Δt is larger than $\Delta t_{\text{free-fall}}$ by two orders of magnitude.

To conclude, according to the cannonball model, a reasonable estimate of detection rate by LIGO-II is about E_{52}^3 per year (assuming a small Δt). The parameter E_{52} is estimated

to be ~ 1 , but there is an uncertainty of almost an order of magnitude. This leads to an uncertainty of order $\sim 10^2$ in the detection rate. The latter will be smaller if Δt is larger than Δt_d .

The maximal observation distance, achieved in the forward direction, is given in Eq. (32). Substituting the above astrophysical parameters, we obtain for the advanced detector

$$R_{\max} = 4 \frac{\gamma m G}{c^2 h_d} \approx 15 \cdot E_{52} \text{ Mpc} . \quad (47)$$

The cosmological redshift is negligible at this distance.

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